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On duality bound methods for nonconvex global optimization

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Abstract A counter-example is given to several recently published results on duality bound methods for nonconvex global optimization.

Keywords Nonconvex global optimization · Duality bound methods

1 Introduction

Consider the following nonconvex global optimization problem

$$\inf\{F(x,y)|\ G(x,y) \leq 0, \quad x \in X,\ \ y \in Y\}, \tag{P}$$

where X is a compact convex subset of $\mathbb{R}^n$, Y a closed convex subset of $\mathbb{R}^p$, $F: X \times Y \to \mathbb{R}, G\colon X \times Y \to \mathbb{R}^m$ and $G(x,y) \leq 0$ means $G_i(x,y) \leq 0,\ i = 1,\ldots,m$, with $G_i(x,y)$ being the ith component of the vector $G(x,y) \in \mathbb{R}^m$.

For solving this problem a branch and bound decomposition algorithm has been proposed in [1, 2, 3, 4, 5] that generates a filter (infinite nested sequence) of partitions sets $M_k, k = 1, 2, \ldots$, such that

$$M_{k+1} \subset M_k \subset \mathbb{R}^m_+, \quad -\infty < \beta(M_k) \leq \min\,(\mathrm{P}), \quad \cap_{i=1}^{+\infty} M_k = \{x^*\}, \tag{1}$$

where

$$\beta(M_k) = \sup_{\lambda \in \mathbb{R}^m_+} \inf_{\substack{y \in Y \\ x \in M_k \cap X}} \{F(x,y) + \langle \lambda, G(x,y) \rangle\} \tag{2}$$

(so $\beta(M)$ is a Lagrangian bound, also called duality bound, for the subproblem $\min\{F(x,y)|\ \ G(x,y) \leq 0,\ x \in M \cap X,\ y \in Y\}$).

The algorithm is said to be *convergent* if $x^* \in X$ and

$$\min\,(\mathrm{P}) = \min\{F(x^*,y)|\ G(x^*,y) \preceq_K 0,\ y \in Y\}, \tag{3}$$

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so that any optimal solution y^* of this problem yields an optimal solution (x^*, y^*) of (P).

In this note we discuss the important issue of under which conditions convergence is guaranteed.

2 A counter-example

The following claims have been proved in [1, 2, 4, 5]:

Claim A (Theorem 3 in [2], Theorem 3.5 in [1] and Theorem 3.1 in [5]) *Convergence holds under the following assumptions:*

(A1) *The functions $F(x,y), G_i(x,y), i = 1,\dots,m$, are continuous on $X \times Y$, while the set X is a compact convex set in $\mathbb{R}^n$ and Y a closed convex set in $\mathbb{R}^p$.*

(A2) *The problem has an optimal solution (so $\beta(M_k) \le \min$ (P) $< +\infty$).*

(A3) *Dual properness at x^* holds, in the sense that*

$$\min\{F(x^*,y)|\ G(x^*,y) \le 0, y \in Y\} = \sup_{\lambda \in \mathbb{R}^m_+} \inf_{y \in Y} \{F(x^*,y) + \langle \lambda, G(x^*,y)\rangle\}.$$

(A4) *There exists a bounded set $Y^0 \subset Y$ such that for every $\lambda \in \mathbb{R}^m_+$ and every $x \in X$ the problem*

$$\min_{y \in Y}\{F(x,y) + \langle \lambda, G(x,y)\rangle\}$$

either has no optimal solution, or it has at least an optimal solution $y \in Y^0$.

Claim B (Theorem 5 in [2], Theorem 3.3 in [1] and Theorem 3.1 in [5]) *Convergence holds under assumptions* (A1), (A2), (A3) *and*

(B4) *The functions $F(x,y), G_i(x,y), i = 1,\dots,m$, are convex in y for every fixed x,*

Regrettably, however, both these claims are contradicted by the following

Proposition *There exists an instance of problem* (P) *satisfying all conditions specified in Claims A and B and such that, nevertheless,*

$$\min\,(\mathrm{P}) < \min\{F(x^*,y)|\ G(x^*,y) \le 0,\ y \in Y\}. \tag{4}$$

Proof Consider the problem

$$\inf\{x\mathrm{e}^{-\sqrt{(1-x)y}}|\ (x+1)/y \le 5,\ 0 \le x \le 1,\ y \ge 1/5\}, \tag{5}$$

which is an instance of (P) with $X = [0,1] \subset \mathbb{R}_+,\ Y = \{y \ge 1/5\},\ K = \mathbb{R}_+,\ F(x,y) = x\mathrm{e}^{-\sqrt{(1-x)y}},\ G(x,y) = (x+1)/y - 5$.

Let $M_k = [1 - 2^{-k}, 1],\ x^* = 1$. For any segment $M \subset [0,1]$ we have

$$\inf_{\substack{y \ge 1/5 \\ x \in M}} \left\{x\mathrm{e}^{-\sqrt{(1-x)y}} + \lambda\left(\frac{x+1}{y} - 5\right)\right\} = -5\lambda,$$

hence $\beta(M) = \sup_{\lambda \ge 0}\{-5\lambda\} = 0\ \forall M \subset [0,1]$, and consequently $\beta(M_k) \le \min$ (P).

We show that for this sequence $\{M_k\}$ and point $x^* = 1$ all conditions (A1), (A2), (A3), (A4), (B4) are satisfied:

(A1), (B4) are obvious;

(A2): Any $(0, \bar{y})$ with $\bar{y} \geq 1/5$ is a feasible solution such that $F(0, \bar{y}) = 0$ while for any feasible solution (x, y) we have $F(x, y) = x\mathrm{e}^{-\sqrt{(1-x)y}} \geq 0$, hence 0 is the optimal value and any $(0, \bar{y})$ with $\bar{y} \geq 1/5$ is an optimal solution.

(A3): Since for $\bar{y} = 1/2$ one has $G(x^*, \bar{y}) = 2(1+1)-5 \leq 4-5 < 0$, Slater regularity condition holds for the convex problem: $\inf \left\{x^*\mathrm{e}^{\sqrt{(1-x^*)y}} \mid (x^*+1)/y \leq 5, y \geq 1/5\right\}$. Therefore, dual properness holds at x^*.

(A4): For $\lambda > 0$ or $0 < x < 1$ the problem

$$\min_{y \geq 1/5} \left\{x\mathrm{e}^{-\sqrt{(1-x)y}} + \lambda\left(\frac{x+1}{y} - 5\right)\right\}$$

has no optimal solution; while for $\lambda = 0, x \in \{0, 1\}$ any $\bar{y} \geq 1/5$ is an optimal solution.

Thus, all specified conditions are fulfilled. Nevertheless, min (P) = 0 while

$$\min\{1.\mathrm{e}^{-\sqrt{(1-1)y}} \mid (1+1)/y \leq 5,\ y \geq 1/5\} = 1,$$

hence (4). □

Conditions (A1) through (A4) and (B4) show that all hypotheses of Theorems 3, 5 in [1], Theorems 3.3, 3.4 in [4], Theorem 1 in [4], and Theorems 3.1, 3.2 in [5], are satisfied, but (4) contradicts the conclusions of each of these theorems. So the basic results in all these papers are erroneous and cannot be used to derive algorithms for problems considered there and elsewhere.

The main error in the proofs given in the mentioned papers is to take it as granted that

$$\sup_{\lambda \geq 0} \inf_{\substack{y \in Y \\ x \in M_k}} \{F(x, y) + \langle \lambda, G(x, y) \rangle\} \underset{k \to +\infty}{\longrightarrow} \sup_{\lambda \geq 0} \inf_{y \in Y} \{F(x^*, y) + \langle \lambda, G(x^*, y) \rangle\} \tag{6}$$

(see e.g. the last argument in the proof of Theorem 3.3 in [1]). In fact if (6) holds then dual properness at x^* will imply

$$\lim_{k \to +\infty} \beta(M_k) = \min_{y \in Y}\{F(x^*, y) \mid G(x^*, y) \leq 0\} \geq \min \text{(P)}$$

from which (3) will follow. Unfortunately, the above example shows that (6) may fail, unless some stronger condition than dual properness is assumed.

For a rigorous foundation of the decomposition method in nonconvex global optimization we refer the reader to [6].

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